Lecture 25

The remainder of the course is on Counting Methods, Discrete Probability and (if time permits) Statistical Distributions.

In this lecture we begin our study of counting methods.

In constructing examples to illustrate methods of counting, we often refer to the tossing of a coin (or of several coins) or to the throwing of a die (or of several dice).

Recall that one side of a coin is called the head, and the other the tail. In contrast, a die has six sides numbered from 1 to 6.

Sum Rule

Let A and B be two experiments. If there are $n(A)$ possible outcomes for A, and $n(B)$ possible outcomes for B, then the total number of possible outcomes for A or B is $n(A) + n(B)$. This requires that A and B have no outcomes in common.

More generally, if $A_1, A_2 \ldots, A_k$ are experiments in which no two of them have any outcomes in common, then the total number of possible outcomes for A_1 or A_2 or ... or A_k is

$$n(A_1) + n(A_2) + \cdots + n(A_k)$$

which we know can be written as

$$\sum_{i=1}^{k} n(A_i).$$

For example, suppose a researcher says to a volunteer, "You can either toss a coin or throw a die, and then report back to me on what the outcome was." The number of possible outcomes is

$$n(A) + n(B) = 2 + 6 = 8$$

where A is "tossing a coin" and B is "throwing a die".

This can be interpreted in the context of Set Theory. If A is one set of outcomes, and B is another set of outcomes, and $A \cap B = \emptyset$, then the number of possible outcomes in A or B is

$$|A \cup B| = |A| + |B| .$$

More generally, if $A_1, A_2, \ldots, A_k$ are sets of outcomes with $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = |A_1| + |A_2| + \cdots + |A_k| .$$

In shorthand notation we write:

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{i=1}^{k} |A_i|$$

What happens if some of the sets are intersecting with each other? Then the formula is more complicated, as we'll see presently.

But now let's consider what happens if the volunteer is asked to throw the die <u>and</u> toss the coin. Let's suppose the coin is tossed first. The outcomes of the two experiments are completely <u>independent</u>, in this sense: whether the volunteer gets a head or a tail has no effect at all on what happens when the die is cast.

So the volunteer has to report back to the researcher giving two separate answers, one for the coin and one for the die.

In this sort of situation, another rule applies.

Product Rule

Let A and B be two independent experiments. That is, the number of choices for B (and the likelihood of each choice) remains the same no matter which choice is made for A.

If there are $n(A)$ possible outcomes for A, and $n(B)$ possible outcomes for B, then the total number of possible outcomes for A <u>and</u> B is $n(A) \cdot n(B)$.

More generally, the number of possible outcomes for A_1 and A_2 and ... and A_k is

$$n(A_1) \cdot n(A_2) \cdots n(A_k) = \prod_{i=1}^{k} n(A_i) .$$

Here we are assuming that the experiments $A_1, A_2, \ldots, A_k$ are pairwise independent events. That is, the outcome of any one of them has no effect on the outcome of any of the others.

Set-Theoretic Interpretations

We've seen that

$$|A \cup B| = |A| + |B|$$

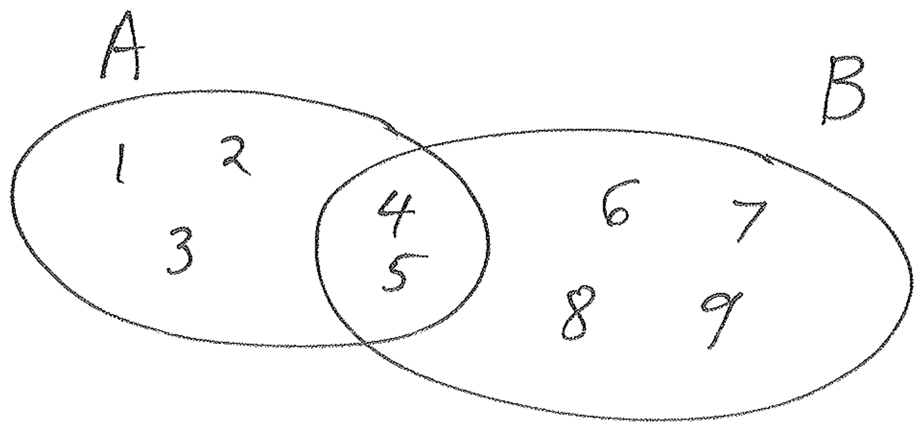
when $A \cap B = \phi$. But what happens if we drop this condition, so that we no longer require A and B to be disjoint?

This leads us to a more general result. Let A and B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

The idea behind this formula is that if we add $|A|$ and $|B|$ then we've counted the elements in $A \cap B$ twice (because they're included in A and they're also included in B).

<u>E.g.</u>



$|A| = 5, \quad |B| = 6, \quad |A \cap B| = 2$

$|A \cup B| = 9 \qquad \left(\text{But } |A| + |B| = 11.\right)$

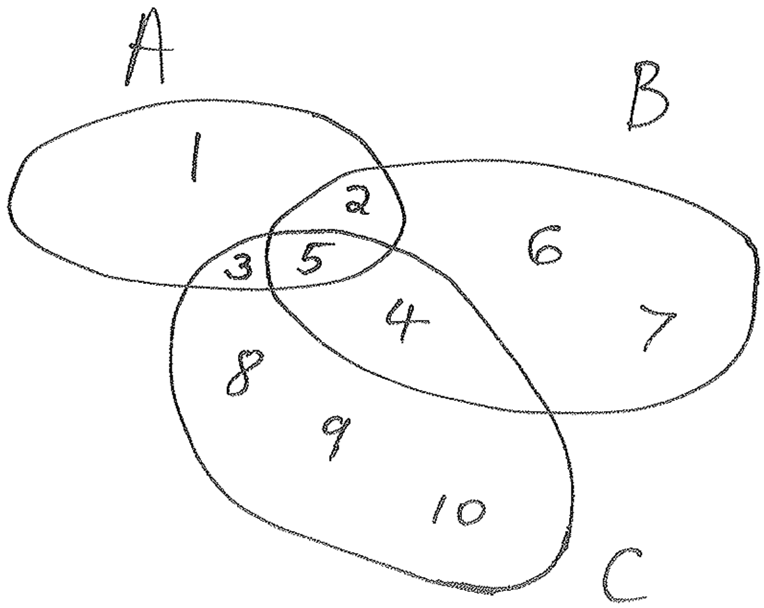
Things get a lot more complicated when we increase the number of sets. Even the union of three sets has a cardinality which is substantially more difficult to compute.

Let A, B and C be finite sets.

Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

<u>E.g.</u>



$|A| = 4$, $|B| = 5$, $|C| = 6$

$|A \cap B| = 2$, $|A \cap C| = 2$, $|B \cap C| = 2$

$|A \cap B \cap C| = 1$

$$|A \cup B \cup C| = (4+5+6) - (2+2+2) + 1$$
$$= 10$$

Exercise

Guess the formula for the cardinality of the union of four finite sets.

The Sum Rule follows directly from the formula for the cardinality of the union of two finite sets, since when A and B are disjoint we get $A \cap B = \phi$ so that $|A \cap B| = 0$.

But what set-theoretic interpretation is available for the Product Rule?

Let's return to the experiment in which the volunteer tosses a coin and casts a die. If the outcomes are H (head) for the coin and 3 for the die, then one very efficient way to report this information is to write down the ordered pair (H, 3).

Of course, all possible combined outcomes can be listed in this way.

But the set of all such ordered pairs is precisely the Cartesian product

$$A \times B.$$

So the Product Rule is nothing more than a restatement (in practical terms) of the formula we already knew for the cardinality of the direct product of two finite sets. If A and B are two sets of independent outcomes, then the number of possible combinations of outcomes in A and B is

$$|A \times B| = |A| \cdot |B| .$$

More generally, for pairwise independent events $A_1, A_2, \dots, A_k$, the number of possible combinations of outcomes for A_1 and A_2 and ... and A_k is

$$|A_1 \times A_2 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdots |A_k| .$$